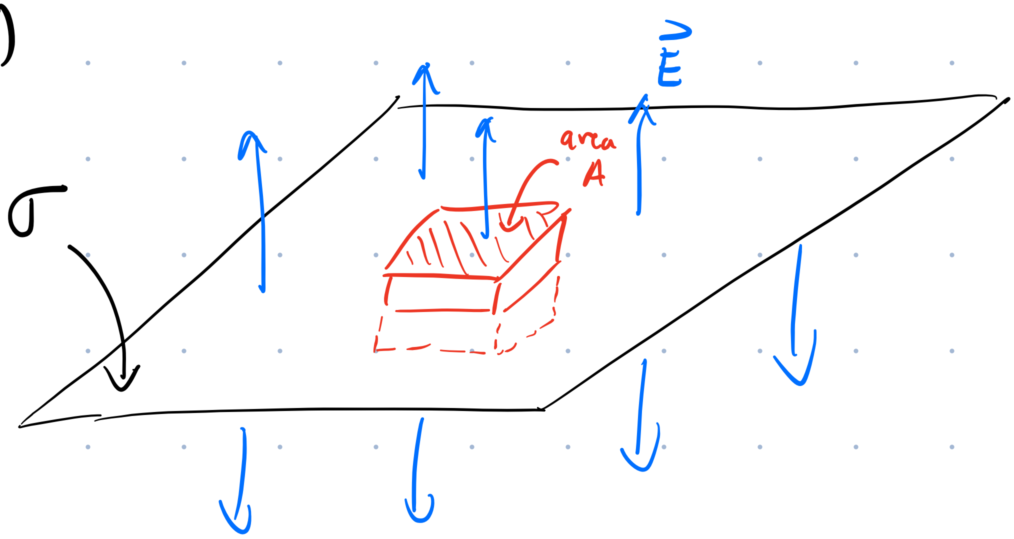
PHYS 301 - Tutorial #4 Group Prob. Sol'ns Sept. 29, 2024

1. (a)



By symmetry, $\vec{E} \perp$ sheet. $\therefore$ $\vec{E}\cdot d\vec{a} \neq 0$ only for the top & bottom of the Gaussian surface.

$$\therefore \oint \vec{E}\cdot d\vec{a} = \int_{top} \vec{E}\cdot d\vec{a} + \int_{btm} \vec{E}\cdot d\vec{a}$$

$\vec{E}\cdot d\vec{a} = E\,da$ for top & btm.

Furthermore, E = const at top & btm parts of Gaussian surface.

$$\therefore \oint \vec{E}\cdot d\vec{a} = E\left[\int_{top} da + \int_{btm} da\right] = 2EA$$

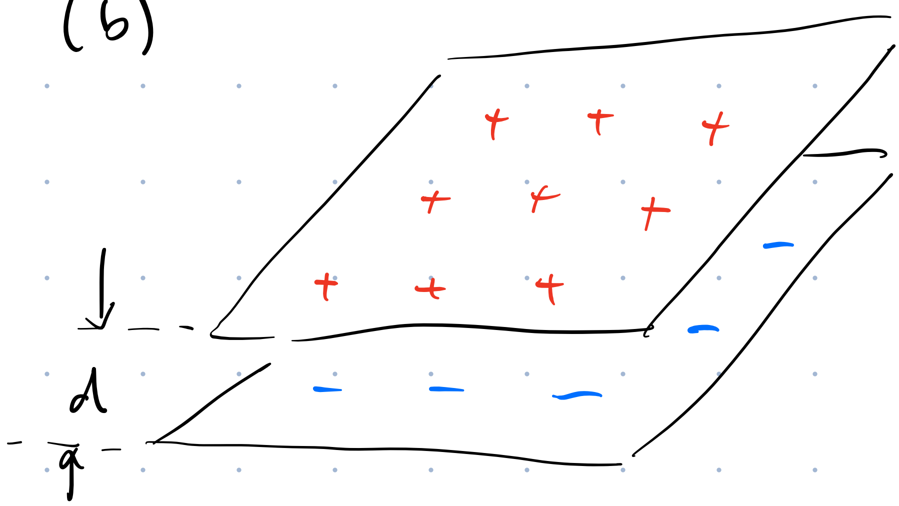
Gauss's Law states that $\oint \vec{E}\cdot d\vec{a} = \frac{Q_{encl}}{\varepsilon_0}$

$Q_{encl} = \sigma A$

$\therefore \; 2E\cancel{A} = \frac{\sigma \cancel{A}}{\varepsilon_0} \qquad \therefore \; \boxed{E = \frac{\sigma}{2\varepsilon_0}}$

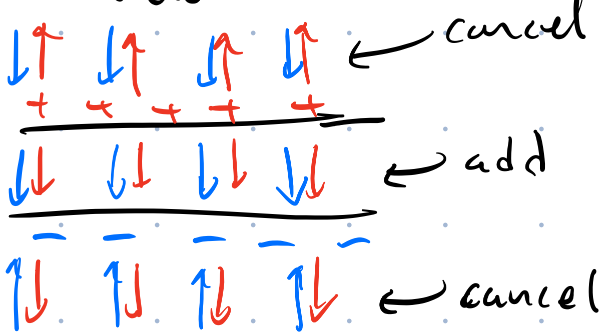
If $\sigma > 0$, $\vec{E}$ points <u>away</u> from the charged sheet

(b)

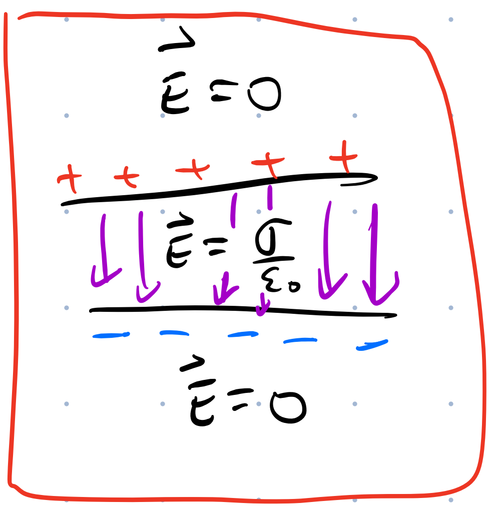


Use the superposition principle to find $\vec{E}_{net}$

Side view



$\Rightarrow$



Between the plates, the two contributions to $\vec{E}_{nt}$ add. They cancel above & below the parallel plates.

(c) In all cases, we found $E_{\parallel} = 0$
(ie. $\vec{E}$ always $\perp$ to charged sheets)

$$\therefore E^{\parallel}_{above} = E^{\parallel}_{below} \quad \checkmark$$

In (a) we found $E^{\perp}_{above} = +\dfrac{\sigma}{2\varepsilon_0}$

$$E^{\perp}_{below} = -\frac{\sigma}{2\varepsilon_0}$$

$$\therefore E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\varepsilon_0} \quad \checkmark$$

In (b) above pos. plate, $E^{\perp}_{above} = 0$ while below the pos. plate $E^{\perp}_{below} = -\dfrac{\sigma}{\varepsilon_0}$

$$\therefore E^{\perp}_{above} - E^{\perp}_{below} = +\frac{\sigma}{\varepsilon_0} \quad \checkmark$$

In (b) above the neg. plate, $E^{\perp}_{above} = -\frac{|\sigma|}{\varepsilon_0}$ charge density of negative plate.

while $E^{\perp}_{below} = 0$

$$\therefore E^{\perp}_{above} - E^{\perp}_{below} = -\frac{|\sigma|}{\varepsilon_0}$$

However, since btm plate is negative,

$$-|\sigma| = \sigma$$

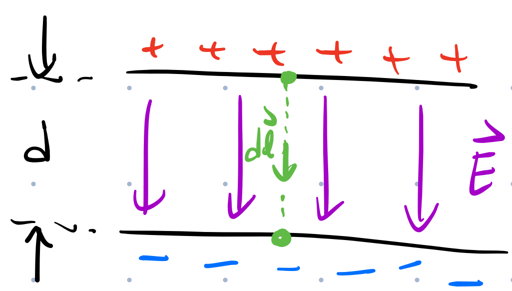
$$\therefore E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\varepsilon_0} \checkmark$$

(d) $\Delta V = -\int \vec{E} \cdot d\vec{l}$

Side view

$\therefore \vec{E} \cdot d\vec{l} = E\, dl$

$E = \frac{\sigma}{\varepsilon_0}$ is const.



$$\therefore \Delta V = -E \int dl = -Ed.$$

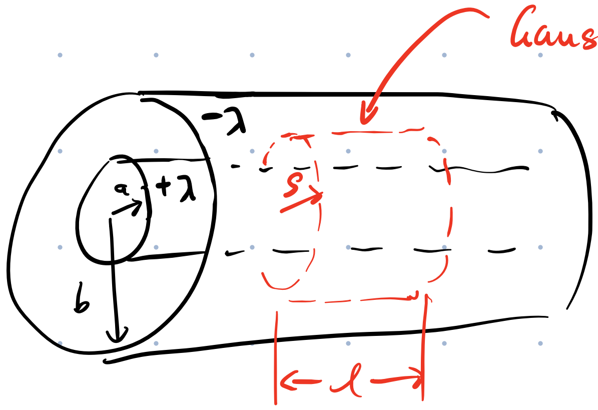
$$\boxed{\therefore \Delta V = V_- - V_+ = -Ed}$$

$\therefore$ As expected, the pos. plate is at a higher pot. than the neg. plate.

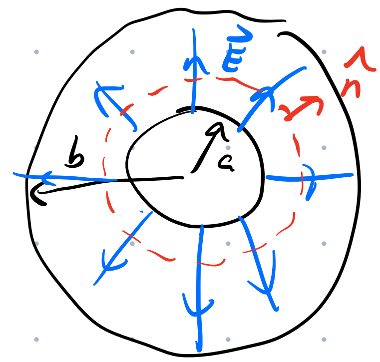
(e) $C = \dfrac{Q}{|\Delta V|} = \dfrac{\sigma A}{E d}$ ← plate area

$\therefore C = \dfrac{\cancel{\sigma} A}{\frac{\cancel{\sigma}}{\varepsilon_0} d} \Rightarrow \boxed{C = \varepsilon_0 \dfrac{A}{d}}$

2. (a) (i)



End View



$Q_{encl} = \lambda \ell$

By symmetry, $\vec{E}$ is radial.

$\therefore \vec{E} \cdot d\vec{a} = 0$ through the left & right sides of Gaussian surface. Get non-zero flux only through the curved surface.

$$\therefore \oint \vec{E} \cdot d\vec{a} = \int_{\text{curved}} \vec{E} \cdot d\vec{a} = \int_{\text{curved}} \overset{\text{const. everywhere on curved surface.}}{E}\, da = E \int_{\text{curved}} da$$

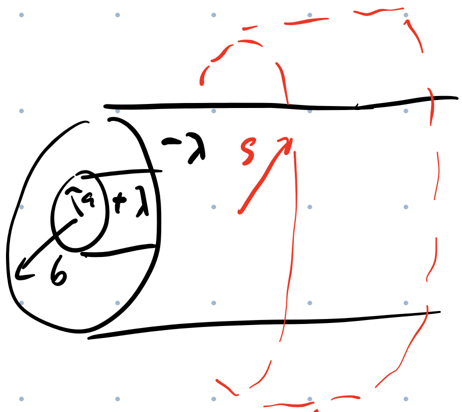
$$\therefore \oint \vec{E}\cdot d\vec{a} = E\, A_{curved} = E\, 2\pi s\, l$$

Since $\oint \vec{E}\cdot d\vec{a} = \dfrac{Q_{encl}}{\varepsilon_0}$,

$$E\, 2\pi s \cancel{l} = \frac{\lambda \cancel{l}}{\varepsilon_0} \Rightarrow \boxed{\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 s}\,\hat{s}}$$

$a < s < b$

(ii)

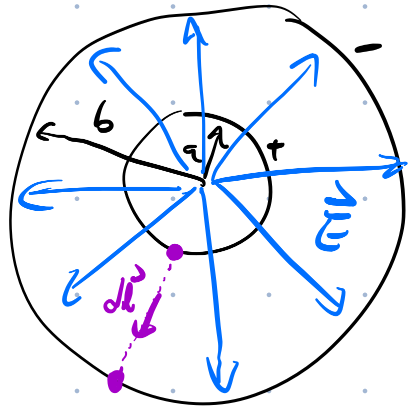


Gaussian surface for (ii)

In this case $Q_{encl} = 0$ (Get equal but opp. contributions from the inner & outer conductors).

$$\therefore \oint \vec{E}\cdot d\vec{a} = 0 \Rightarrow \boxed{\vec{E} = 0} \quad s > b$$

(b) End view



$d\vec{l} = ds\,\hat{s}$

$$\Delta V = -\int \vec{E}\cdot d\vec{l}$$

$$\vec{E}\cdot d\vec{l} = E\,ds$$

$$\therefore \Delta V = -\int_{s=a}^{b} \frac{\lambda}{2\pi\varepsilon_0 s}\,ds = -\frac{\lambda}{2\pi\varepsilon_0}\int_a^b \frac{ds}{s}$$

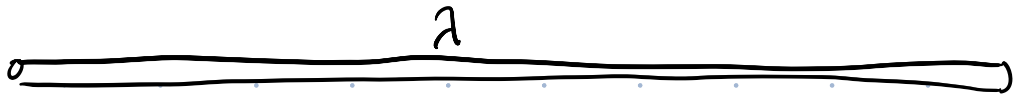
$$\boxed{\therefore \Delta V = V_- - V_+ = -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{b}{a}\right)}$$

(c)

$$C = \frac{q}{|\Delta V|} = \frac{\cancel{q}^{\,1}}{\frac{\cancel{q}/l}{2\pi\varepsilon_0}\ln\left(\frac{b}{a}\right)}$$

$$\therefore C = \frac{2\pi\varepsilon_0 l}{\ln\left(\frac{b}{a}\right)} \Rightarrow \boxed{C_l = \frac{C}{l} = \frac{2\pi\varepsilon_0}{\ln\left(\frac{b}{a}\right)}}$$

3(a)



Know $\Delta V = V_s - V_a = -\int_a^s \vec{E} \cdot d\vec{l}$

If we arbitrarily set $V_a = 0$, the $V_s = V(s)$ can be expressed as:

$$V(s) = -\int_a^s \vec{E} \cdot d\vec{l}$$

$d\vec{l} = ds\,\hat{s}$ s.t. $\vec{E} \cdot d\vec{l} = \dfrac{\lambda\, ds}{2\pi\varepsilon_0 s}$

$$\therefore V(s) = -\frac{\lambda}{2\pi\varepsilon_0}\int_a^s \frac{ds'}{s'} = \boxed{-\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{s}{a}\right)}$$

(b) $\vec{E} = -\vec{\nabla} V$

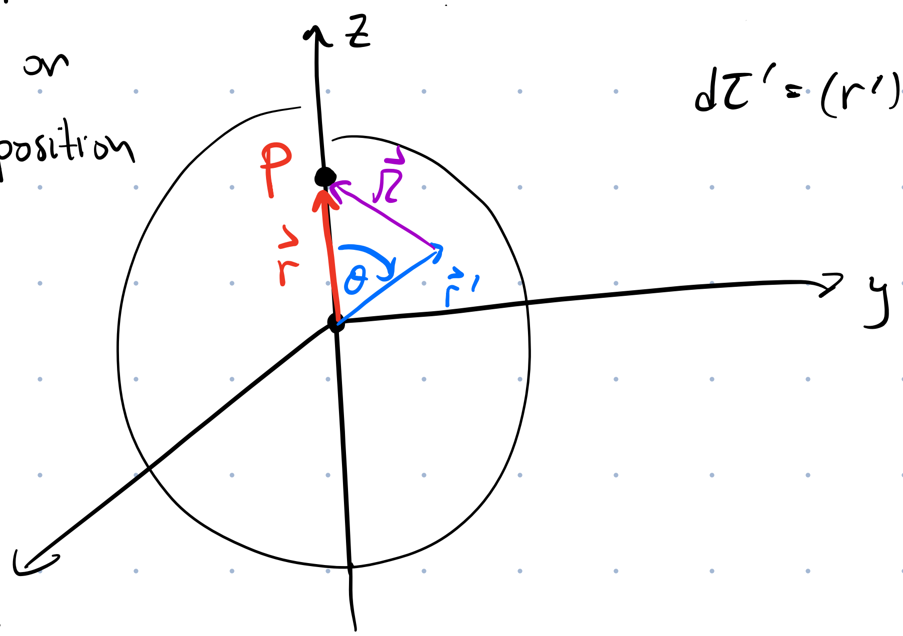
$$\therefore \vec{E} = -\frac{d}{ds}\left(-\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{s}{a}\right)\right)\hat{s} = \boxed{\frac{\lambda}{2\pi\varepsilon_0 s}\hat{s}}$$ as expected.

4(a) $$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,')}{\mathscr{r}}\,d\tau'$$

(i) $r < R$ $\quad \rho(\vec{r}\,') = \rho$ (const).

Place origin at centre of sphere

Arbitrarily place field pt. P on z axis at position z.



$d\tau' = (r')^2 \sin\theta \, d\theta \, d\phi \, dr'$

In this case, $\vec{r} = z\hat{z}$

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,' \Rightarrow \mathscr{r}^2 = \vec{\mathscr{r}}\cdot\vec{\mathscr{r}} = z^2 + (r')^2 - 2zr'\cos\theta$$

$$\therefore \mathscr{r} = \sqrt{z^2 + (r')^2 - 2zr'\cos\theta}$$

$$\therefore V = \frac{\rho}{4\pi\varepsilon_0}\int \frac{(r')^2 \sin\theta \, d\theta \, d\phi \, dr'}{\sqrt{z^2 + (r')^2 - 2zr'\cos\theta}}$$

Do the ϕ integral

$$V = \frac{\rho}{2\varepsilon_0} \int_{\theta=0}^{\pi} \int_{r'=0}^{R} \frac{(r')^2 \sin\theta \, d\theta \, dr'}{\sqrt{z^2+(r')^2-2zr'\cos\theta}}$$

Do the θ integral

$$u = z^2+(r')^2-2zr'\cos\theta$$

$$du = 2zr'\sin\theta \, d\theta \quad \Rightarrow \quad \sin\theta \, d\theta = \frac{du}{2zr'}$$

$$\theta = 0, \quad u = z^2+(r')^2-2zr' \equiv u_0$$

$$\theta = \pi, \quad u = z^2+(r')^2+2zr' \equiv u_\pi$$

$$V = \frac{\rho}{4\varepsilon_0} \int_{r'=0}^{R} \int_{u=u_0}^{u_\pi} \frac{r'}{z} u^{-1/2} \, du \, dr'$$

$$= \frac{\rho}{4\varepsilon_0} 2 \int_{r'=0}^{R} \frac{r'}{z} \left(u_\pi^{1/2} - u_0^{1/2} \right) dr'$$

$$= \frac{\rho}{2\varepsilon_0} \int_{r'=0}^{R} \frac{r'}{z} \left[\underbrace{\sqrt{z^2+(r')^2+2zr'}}_{z+r'} - \underbrace{\sqrt{z^2+(r')^2-2zr'}}_{|z-r'|} \right] dr'$$

$$(z-r')^2 = z^2 + (r')^2 - 2zr' \Rightarrow u_0^{1/2} = \begin{cases} z - r' & z > r' \\ r' - z & z < r' \end{cases}$$

$$(z+r')^2 = z^2 + (r')^2 + 2zr' \Rightarrow u_\pi^{1/2} = z + r'$$

$$\therefore\ u_\pi^{1/2} - u_0^{1/2} = \begin{cases} z + r' - (z - r') = 2r' & z > r' \\ z + r' - (r' - z) = 2z & z < r' \end{cases}$$

$$\therefore\ V = \frac{\rho}{2\varepsilon_0}\left[\underbrace{\int_{r'=0}^{z} \frac{r'}{z} 2r' \, dr'}_{z > r'} + \underbrace{\int_{r'=z}^{R} \frac{r'}{z} 2z \, dr'}_{z < r'}\right]$$

$$= \frac{\rho}{\varepsilon_0}\left[\frac{1}{z}\int_{r'=0}^{z} (r')^2 dr' + \int_{r'=z}^{R} r' \, dr'\right]$$

$$= \frac{\rho}{\varepsilon_0}\left[\frac{1}{z}\frac{z^3}{3} + \frac{1}{2}\left(R^2 - z^2\right)\right]$$

$$= \frac{\rho}{\varepsilon_0}\left[\frac{z^2}{3} + \frac{R^2}{2} - \frac{z^2}{2}\right] = \frac{\rho}{6\varepsilon_0}\left[2z^2 + 3R^2 - 3z^2\right]$$

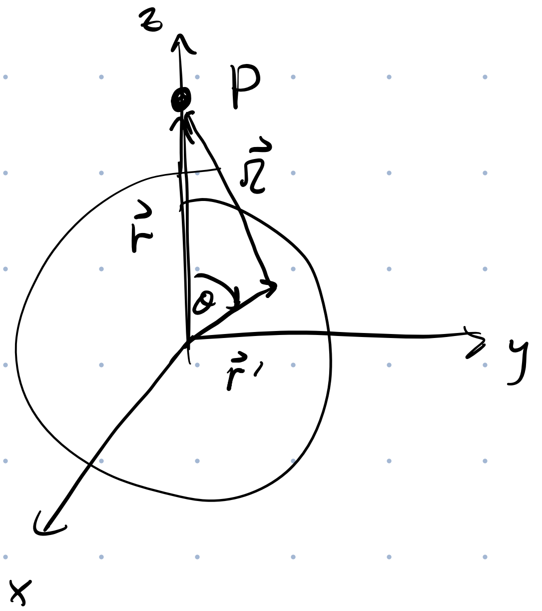
$$\therefore V = \frac{\rho}{6\varepsilon_0}\left[3R^2 - z^2\right] = \boxed{\frac{\rho R^2}{6\varepsilon_0}\left[3 - \left(\frac{z}{R}\right)^2\right]}$$

Subbing in $\rho = \dfrac{q}{\frac{4}{3}\pi R^3}$

$$V = \frac{3q}{6 \cdot 4\pi R\varepsilon_0}\left[3 - \left(\frac{z}{R}\right)^2\right]$$

$$\boxed{\therefore V = \frac{q}{8\pi\varepsilon_0 R}\left[3 - \left(\frac{z}{R}\right)^2\right]}$$

(ii) For a pt. $z > R$



still have

$$\mathcal{R}^2 = \vec{\mathcal{R}} \cdot \vec{\mathcal{R}} = z^2 + (r')^2 - 2zr'\cos\theta$$

Everything is the same as in (i), except we never have the case $z < r'$.

i.e. $|z - r'| = z - r'$

$$\therefore u_\pi^{1/2} - u_0^{1/2} = z + r' - (z - r') = 2r' \text{ always.}$$

$$\therefore V = \frac{\rho}{2\varepsilon_0} \int_{r'=0}^{R} \frac{r'}{z} 2r' \, dr'$$

$$= \frac{\rho}{\varepsilon_0 z} \int_{r'=0}^{R} (r')^2 dr' = \frac{\rho}{\varepsilon_0 z} \frac{R^3}{3} \left(\frac{4\pi}{4\pi}\right)$$

$$\therefore V = \frac{1}{4\pi\varepsilon_0} \frac{\overbrace{\rho \frac{4}{3}\pi R^3}^{q}}{z}$$

$$\boxed{\therefore V = \frac{1}{4\pi\varepsilon_0} \frac{q}{z} \quad \text{like a pt. charge @ sphere centre!}}$$

(b) Start w/ (ii)

$\vec{E} = -\vec{\nabla} V$ write $\frac{1}{z}$ as $\frac{1}{r}$ for this calc.

$\therefore \vec{E} = -\hat{r}\frac{\partial}{\partial r}\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r}\right) = \boxed{\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}}$ like pt. charge ✓

$r > R$

(i) $V = \frac{q}{8\pi\varepsilon_0 R}\left[3 - \left(\frac{z}{R}\right)^2\right]$

again, write $z = r$

$$\vec{E} = -\vec{\nabla} V = -\hat{r}\frac{\partial}{\partial r}\left(\frac{q}{8\pi\varepsilon_0 R}\left[3 - \left(\frac{r}{R}\right)^2\right]\right)$$

$$= \frac{q}{8\pi\varepsilon_0 R}\frac{2r}{R^2}\hat{r}$$

$$\boxed{\therefore \vec{E} = \frac{qr}{4\pi\varepsilon_0 R^3}\hat{r} \qquad r < R}$$

(c) $$W = \frac{1}{2}\int \rho V d\tau$$

since $\rho = 0$ for $r > R$, use potential inside sphere only.

$$W = \frac{\rho}{2}\int \frac{q}{8\pi\varepsilon_0 R}\left[3-\left(\frac{r}{R}\right)^2\right] d\tau$$

$$= \frac{\rho q}{16\pi\varepsilon_0 R} \underbrace{\int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi} \sin\theta \, d\theta}_{4\pi} \int_{r=0}^{R} \left[3-\left(\frac{r}{R}\right)^2\right] r^2 dr$$

$$= \frac{\rho q}{4\varepsilon_0 R} \underbrace{\left[R^3 - \frac{1}{5}R^3\right]}_{\frac{4}{5}R^3} = \frac{\rho q R^2}{5\varepsilon_0}$$

sub. $\rho = \frac{q}{\frac{4}{3}\pi R^3}$ $\Rightarrow$ $\boxed{W = \frac{3q^2}{20\pi\varepsilon_0 R}}$

5. Expect $\nabla^2 V = -\rho/\varepsilon_0$.

$$\therefore \nabla^2 V = \nabla^2 \left[\frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}\,')}{\mathscr{r}} \, d\tau' \right]$$

The ∇ operator acts on $\vec{r}$, not $\vec{r}\,'$

$$\therefore \nabla^2 V = \frac{1}{4\pi\varepsilon_0} \int \rho(\vec{r}\,') \underbrace{\nabla^2\left(\frac{1}{\mathscr{r}}\right)}_{-4\pi\delta^3(\vec{\mathscr{r}})} d\tau'$$

$$= -\frac{1}{\varepsilon_0} \int \rho(\vec{r}\,') \, \delta^3(\vec{\mathscr{r}}) \, d\tau'$$

δ^3 fcn selects $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,' = 0$

$\therefore \vec{r}\,' = \vec{r}$.

$$\boxed{\therefore \nabla^2 V = -\frac{1}{\varepsilon_0} \rho(\vec{r})}$$ ✓